Phase and Scaling Properties of Determinants Arising in Topological Field Theories[1]

David H. Adams[2] and Siddhartha Sen

School of Mathematics, Trinity College, Dublin 2, Ireland.

email: dadams,sen@maths.tcd.ie

Abstract

In topological field theory determinants of maps with negative as well as positive eigenvalues arise. We give a generalisation of the zeta-regularisation technique to derive expressions for the phase and scaling-dependence of these determinants. For theories on odd-dimensional manifolds a simple formula for the scaling dependence is obtained in terms of the dimensions of cohomology spaces. This enables a non-perturbative feature of Chern-Simons gauge theory to be reproduced by semiclassical methods

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Topological field theories (TFTs) are of interest because they provide examples of quantum field theories which are exactly solvable and because they provide a new way of looking at topological invariants of manifolds [2],[3]. A particular TFT, the Chern-Simons gauge theory on 3-dimensional manifolds, has led to new invariants [4],[5]. (For a review of TFTs see [1]).

In topological field theory, given a topological action functional $S(\omega)$ for fields ω on a manifold M , an object of interest is the partition function

$$Z(\beta) = \int_{\Gamma} \mathcal{D}\omega \, e^{-\beta S(\omega)} \tag{1}$$

where the formal integration is over the infinite-dimensional vectorspace Γ of fields ω . We have included in (1) a scaling parameter β which we allow to be complex-valued. (Typically β is either real or purely imaginary; it is often taken to be a constant equal to 1 or $-i$). For the cases we consider in this paper the manifold M is required to be compact, without boundary and oriented (e.g. a sphere of arbitrary dimension).

For a wide class of TFTs where the action $S(\omega)$ is quadratic (see (16) below for a specific example) the partition function can be formally evaluated by the method of A. Schwarz [2],[6]. This leads to an expression for (1) consisting of a product of determinants of certain maps associated with $S(\omega)$. One of these determinants is[3]

$$det(\beta\widetilde{T})^{-1/2} \tag{2}$$

where $\widetilde{T}$ is obtained by discarding the zero-modes of the selfadjoint map T on Γ given by

$$S(\omega) = < \omega \,, T\omega > \ . \tag{3}$$

The inner product $< \cdot , \cdot >$ in Γ used to obtain T from $S(\omega)$ in (3) is constructed from a Euclidean metric on M (as in [6, p.437]). The other determinants in the expression for the partition function appear because of the zero-modes of T . They are all real-valued and do not involve the parameter β . Hence the phase of the

[3]This should really be $det(\beta\frac{1}{\pi}\widetilde{T})^{-1/2}$ since $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. However the numerical factor $1/\pi$ in the determinant is usually considered to be irrelevant and discarded.

partition function (1) and its dependence on the scaling parameter β are determined solely by the determinant (2). The determinants in the expression for the partition function are determinants of maps on infinite-dimensional vectorspaces and must therefore be regularised in order to obtain a finite expression. This is done using the zeta-regularisation technique.

In this paper we consider a subtlety in the zeta-regularisation of the determinant (2). The zeta-regularisation technique requires the map to be positive, i.e. all its eigenvalues must be positive. But the action functional $S(\omega)$ of the TFT typically takes negative as well as positive values, so from (3) it follows that $\widetilde{T}$ has negative as well as positive eigenvalues. This problem was sidestepped by Schwarz (and in most subsequent work on TFTs) by replacing $\widetilde{T}$ in (2) by the positive map $|\widetilde{T}|$. This map is defined in the following way: Take a basis $\{\omega_j\}$ for Γ of eigenvectors of T with eigenvalues $\{\lambda_j\}$, then $|T|$ is defined by setting $|T|\omega_j = |\lambda_j|\omega_j$ and $|\widetilde{T}|$ is obtained from $|T|$ by discarding the zero-modes.

For a particular case (with $\beta = i$) E. Witten has shown in [5, §2] how the zeta-regularisation technique can be generalised to evaluate (2) (see also [7, §7.2]). He found that a complex phase factor appears, determined by $\eta(0\,;T)$, where $\eta(s\,;T)$ is the eta-function of T. In this paper we evaluate the determinant (2) (with arbitrary $\beta \in \mathbf{C}$) for all the above-mentioned TFTs considered by Schwarz. This is done using a straightforward generalisation of the usual zeta-regularisation technique and analytic continuation in β, and generalises the calculation of Witten mentioned above. The following expression is obtained: Let $\mathbf{C}_+$ and $\mathbf{C}_-$ denote the upper and lower halfplanes of $\mathbf{C}$ respectively, then for $\beta = |\beta|e^{i\theta} \in \mathbf{C}_\pm$ with $\theta \in [-\pi\,,\pi]$ we find

$$det(\beta\widetilde{T})^{-1/2} = e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi}\mp 1)\zeta\,\pm\,\eta)}\,|\beta|^{-\zeta/2}\,det(|\widetilde{T}|)^{-1/2} \tag{4}$$

where ζ and η are the analytic continuations to $s=0$ of the zeta-function $\zeta(s\,;|T|)$ and eta-function $\eta(s\,;T)$ respectively (defined as in (7) and (10) below) and $det(|\widetilde{T}|)^{-1/2}$ is defined by the usual zeta-regularisation technique. In particular, for $\lambda \in \mathbf{R}_+$ we get

$$det(\lambda\widetilde{T})^{-1/2} = e^{\pm\frac{i\pi}{4}(\zeta-\eta)}\,\lambda^{-\zeta/2}\,det(|\widetilde{T}|)^{-1/2} \tag{5}$$

and

$$det(i\lambda\widetilde{T})^{-1/2} = e^{-\frac{i\pi}{4}\eta}\,\lambda^{-\zeta/2}\,det(|\widetilde{T}|)^{-1/2}\;. \qquad (6)$$

Note that for $\beta \in \mathbf{R}$ there is a phase ambiguity in (4) (analogous to the ambiguity in $\sqrt{-1} = \pm i$) while there is no ambiguity for $\beta \in \mathbf{C} - \mathbf{R}$ (e.g. when β is purely imaginary). It is not immediately obvious that the phase and scaling factors in (4)–(6) are finite, since this requires the zeta-function $\zeta(s\,;\,|T|)$ and eta-function $\eta(s\,;\,T)$ to have analytic continuations regular at $s = 0$. If T were elliptic then this would follow from standard results in mathematics; however for the cases arising in TFTs the map T in (3) is *not* elliptic. We will nevertheless show below that $\zeta(s\,;\,|T|)$ and $\eta(s\,;\,T)$ do in fact have analytic continuations regular at $s = 0$, so the expressions (4)–(6) are finite. (We do not claim that this is a new mathematical result, but for the sake of completeness we give a simple derivation). We also derive a simple formula for $\zeta = \zeta(0\,;\,|T|)$ in terms of the dimensions of certain cohomology spaces when M has odd dimension ((23) below). This leads to a simple expression for the scaling dependence of (4)–(6).

Determinants of the form (2) are also relevant for TFTs where the action $S(\omega)$ contains higher order terms as well as the quadratic term. In this case determinants of the form (2) appear in the semiclassical approximation for the partition function of the theory. A particular TFT with non-quadratic action is the Chern-Simons gauge theory on 3-dimensional manifolds (given by (28) below), which was shown to be solvable by E. Witten in [5]. We will discuss below how the dependence of the semiclassical approximation on the parameter k in this theory can be obtained from our calculation of (2). Because it is a solvable theory for a field with self-interactions the Chern-Simons gauge theory provides a “mathematical laboratory” in which predictions of perturbation theory can be tested. A basic prediction of perturbative quantum field theory is that the semiclassical approximation should coincide with the non-perturbative expression for the partition function in the limit where the parameter k of the theory becomes large. The large k limit of the partition function, with gauge group $SU(2)$, has been explicitly calculated by Witten's non-perturbative method for a large number of 3-manifolds in a program initiated by

D. Freed and R. Gompf [8],[9]. They found that the $k-$dependence of the partition function in this limit is given by a simple expression ((32) below). Subsequent work by L. Jeffrey [10] and L. Rozansky [11] has verified this expression for large classes of 3-manifolds. The expression we obtain below for the $k-$dependence of the semiclassical approximation turns out to be identical to this non-perturbative expression. Thus we reproduce a non-perturbative feature of the Chern-Simons gauge theory from perturbation theory.

Before evaluating (2) we briefly recall the usual zeta-regularisation technique. The zeta-function of a positive selfadjoint linear map A is defined by

$$\zeta(s\,;\,A)=\sum_j \frac{1}{\lambda_j^s} \qquad\qquad s\in\mathbf{C} \tag{7}$$

where $\{\lambda_j\}$ are the non-zero eigenvalues of A (so $\lambda_j>0$ for all λ_j in (7)) with each eigenvalue appearing the same number of times as its multiplicity. With $\widetilde{A}$ obtained from A by discarding the zero-modes we can formally write

$$det(\widetilde{A})=\prod_j \lambda_j = e^{-\zeta'(0\,;\,A)}\,. \tag{8}$$

When A acts on an infinite-dimensional vectorspace $\zeta(s\,;\,A)$ is divergent around $s=0$. However in many cases of interest it turns out that $\zeta(s\,;\,A)$ is well-defined for $Re(s)>>0$ and extends by analytic continuation to a meremorphic function on $\mathbf{C}$ which is regular at $s=0$. Then we can use the analytic continuation of $\zeta(s\,;\,A)$ to give well-defined meaning to the r.h.s. of (8) and use this to define $det(\widetilde{A})$ in (8). For $\beta\in\mathbf{R}_+$ we then obtain a well-defined expression for $det(\beta\widetilde{A})$ by replacing $\widetilde{A}$ by $\beta\widetilde{A}$ in (8). This leads to

$$det(\beta\widetilde{A})=\beta^{\zeta(0\,;\,A)}\,e^{-\zeta'(0\,;\,A)}\,. \tag{9}$$

Using (9) we can define $det(\beta\widetilde{A})$ for arbitrary $\beta\in\mathbf{C}$ via analytic continuation in β. To do this we must fix a convention for defining z^a for $z\in\mathbf{C}$ and $a\in\mathbf{R}$. The natural way to do this is to write $z=|z|e^{i\theta}$ with $\theta\in[-\pi\,,\pi]$ and set $z^a=|z|^a e^{i\theta a}$. This is well-defined for all $a\in\mathbf{R}$ provided $z\notin\mathbf{R}_-$; if $z\in\mathbf{R}_-$ then there is a phase ambiguity. With this convention (9) is defined for all $\beta\in\mathbf{C}$ up to a phase ambiguity

for $\beta \in \mathbf{R}_-$. Finally, recall that the eta-function of a selfadjoint linear map B (which may have both positive and negative eigenvalues) is defined by

$$\eta(s\,;\,B) = \sum_k \frac{1}{(\lambda_k^{(+)})^s} - \sum_l \frac{1}{(-\lambda_l^{(-)})^s} \tag{10}$$

where $\{\lambda_k^{(+)}\}$ and $\{\lambda_l^{(-)}\}$ are the strictly positive- and strictly negative eigenvalues of B respectively. In many cases of interest it turns out that $\eta(s\,;\,B)$ is well-defined for $Re(s) >> 0$ and extends by analytic continuation to a meremorphic function on $\mathbf{C}$ which is regular at $s=0$.

We shall now evaluate the determinant (2). Formally we have

$$det(\beta\widetilde{T})^{-1/2} = (\,det(\beta T_+)\,det(\beta T_-)\,)^{-1/2} \tag{11}$$

where T_+ and T_- are obtained from T by restricting to the strictly positive- and strictly negative modes respectively. Note that $-T_-$ is positive (i.e. has positive eigenvalues) and that

$$\zeta(s\,;\,|T|) \;=\; \zeta(s\,;\,T_+) + \zeta(s\,;\,-T_-) \tag{12}$$

$$\eta(s\,;\,T) \;=\; \zeta(s\,;\,T_+) - \zeta(s\,;\,-T_-) \tag{13}$$

From (11), using (8), (9) and (12) we get

$$\begin{aligned} det(\beta\widetilde{T})^{-1/2} &= det(\beta T_+)^{-1/2}\,det((-\beta)\,(-T_-))^{-1/2} \\ &= \beta^{-\zeta(0\,;\,T_+)/2}\,(-\beta)^{-\zeta(0\,;\,-T_-)/2}\,e^{(\zeta'(0\,;\,T_+)+\zeta'(0\,;\,-T_-))/2} \\ &= \beta^{-\zeta(0\,;\,T_+)/2}\,(-\beta)^{-\zeta(0\,;\,-T_-)/2}\,det(|\widetilde{T}|)^{-1/2} \end{aligned} \tag{14}$$

For $\beta = |\beta|e^{i\theta} \in \mathbf{C}_\pm$ with $\theta \in [-\pi\,,\pi]$ we have $-\beta = |\beta|e^{i(\theta\mp\pi)}$ with $\theta \mp \pi \in [-\pi\,,\pi]$ and a simple calculation using (12) and (13) shows

$$\beta^{-\zeta(0\,;\,T_+)/2}\,(-\beta)^{-\zeta(0\,;\,-T_-)/2} = e^{-\frac{i\pi}{4}((\frac{2\theta}{\pi}\mp 1)\zeta(0\,;\,|T|)\,\pm\,\eta(0\,;\,T))}\,. \tag{15}$$

Substituting this in (14) gives (4).

As pointed out previously, for the expression (4) to have well-defined meaning $\zeta(s\,;\,|T|)$ and $\eta(s\,;\,T)$ must be regular at $s=0$. We will now show that this is the

case for the cases of interest in TFT. In doing so we derive a simple formula for $\zeta(0\,;\,|T|)$ when M has odd dimension. For the sake of concreteness we will work with a specific topological action functional

$$S(\omega)=\int_M \omega\wedge d_m\omega\,. \tag{16}$$

The fields ω are the real-valued differential forms on M of degree m and d_q denotes the exterior derivative on $q-$forms. M is required to have odd dimension $n=2m+1$ and we assume that m is odd, since for m even (16) is identically zero. The quadratic action functionals in other TFTs are generalisations of (16) and it is easily checked that the following arguments continue to hold for these. A choice of metric on M enables us to construct an inner product in the space of differential forms in the usual way (as in [6, p.437]) and with this we can write

$$S(\omega)=<\omega\,,T\omega> \quad , \quad T=*d_m \tag{17}$$

where $*$ is the Hodge star-map (as in [6, p.437]). We denote the space of $q-$forms on M by $\Omega^q(M)$ and define the Laplace-operator on $\Omega^q(M)$ by

$$\Delta_q=d_q^*d_q+d_{q-1}d_{q-1}^* \quad , \quad q=0,1,\dots,n \tag{18}$$

(with $d_{-1}=d_n=0$). We will derive a relationship between the zeta-function of $|T|$ and the zeta-functions of $\Delta_0\,,\,\Delta_1\,,\dots,\,\Delta_m\,.$ To do this we will use the following simple observation: Consider linear maps A and B on a vectorspace, satisfying $AB=BA=0\,.$ Then if $\{\lambda_j\}$ denotes the collection of non-zero eigenvalues of $A+B$ (with each eigenvalue appearing the same number of times as its multiplicity) we have

$$\{\lambda_j\}=\{\lambda'_k\}\cup\{\lambda''_l\} \tag{19}$$

where $\{\lambda'_k\}$ and $\{\lambda''_l\}$ are the non-zero eigenvalues of A and B respectively. (This is an elementary fact in linear algebra which is easily verified). Setting $A=d_q^*d_q$ and $B=d_{q-1}d_{q-1}^*$ the property $AB=BA=0$ follows from $d_qd_{q-1}=0\,$, and it follows from (18) and (19) that

$$\begin{aligned}\zeta(s\,;\,\Delta_q) &= \zeta(s\,;\,d_q^*d_q)+\zeta(s\,;\,d_{q-1}d_{q-1}^*)\\ &= \zeta(s\,;\,d_q^*d_q)+\zeta(s\,;\,d_{q-1}^*d_{q-1})\end{aligned} \tag{20}$$

where we have used the simple fact that for any linear map C the maps C^*C and CC^* have the same non-zero eigenvalues. A simple induction argument based on (20) and starting with $\zeta(s\,;\,d_m^*d_m)=\zeta(s\,;\,\Delta_m)-\zeta(s\,;\,d_{m-1}^*d_{m-1})$ shows that

$$\zeta(s\,;\,d_m^*d_m)=(-1)^m\sum_{q=0}^{m}(-1)^q\zeta(s\,;\,\Delta_q)\,. \tag{21}$$

The map T in (17) has the property $T^2=d_m^*d_m$ and from the definition (7) we see that $\zeta(s\,;\,T^2)=\zeta(2s\,;\,|T|)$. It follows from (21) that

$$\zeta(s\,;\,|T|)=(-1)^m\sum_{q=0}^{m}(-1)^q\zeta(\frac{s}{2}\,;\,\Delta_q)\,. \tag{22}$$

This shows that $\zeta(s;|T|)$ is well-defined for $Re(s)>>0$ with analytic continuation regular at $s=0$, since the zeta-functions of the Laplace-operators Δ_q are known to have these properties (see e.g. [12, ch.28]). When $dimM$ is odd we have $\zeta(0\,;\,\Delta_q)=-dimH^q(d)$ (see [12, ch.28]), where $H^q(d)=ker(d_q)\big/Im(d_{q-1})$ is the q'th cohomology space of d. It follows from (22) that in this case

$$\zeta(0\,;\,|T|)=(-1)^{m+1}\sum_{q=0}^{m}(-1)^q dimH^q(d)\,. \tag{23}$$

We now consider the eta-function $\eta(s\,;\,T)$. A standard result in elliptic operator theory states that the eta-function of an elliptic selfadjoint map is regular at $s=0$. (This is due to M. Atiyah, V. Patodi and I. Singer [13] in the case where $dimM$ is odd, and P. Gilkey [14] when $dimM$ is even). The map T in (17) is selfadjoint but not elliptic. However we can construct an elliptic selfadjoint map D such that $\eta(s\,;\,D)=\eta(s\,;\,T)$, from which it follows that $\eta(s\,;\,T)$ is regular at $s=0$. For $q=0,1,\dots,m$ we extend d_q to a map on $\oplus_{q=0}^m\Omega^q(M)$ by setting $d_q=0$ on $\Omega^p(M)$ for $p\neq q$. We define the map $\widetilde{D}$ on $\oplus_{q=0}^m\Omega^q(M)$ by $\widetilde{D}=\sum_{q=0}^m(d_q+d_q^*)$ and set $D=T+\widetilde{D}$, with T as in (17). D is clearly selfadjoint and a simple calculation using the property $d_qd_{q-1}=0$ shows that $D^2=\sum_{q=0}^m\Delta_q$, which is elliptic, so D is elliptic. It is immediate from the definitions of $\widetilde{D}$ and T that $T\widetilde{D}=\widetilde{D}T=0$ and it follows from (19) that

$$\eta(s\,;\,D)=\eta(s\,;\,T)+\eta(s\,;\,\widetilde{D})\,. \tag{24}$$

To show $\eta(s\,;\,D)\ =\ \eta(s\,;\,T)$ we must show that $\eta(s\,;\,\widetilde{D})\ =\ 0$. We consider the eigenvalue equation $\widetilde{D}\omega=\lambda\omega$ with $\omega=\oplus_{q=0}^{m}\omega_q\in\oplus_{q=0}^{m}\Omega^q(M)$. This is equivalent to the collection of equations

$$d_q\omega_q+d_{q+1}^*\omega_{q+2}=\lambda\omega_{q+1}\quad,\quad q=0,1,\ldots,m-1 \tag{25}$$

(with $\omega_{m+1}\ =\ 0$). If ω is a solution to (25) then we set $\omega'\ =\ \oplus_{q=0}^{m}\omega_q'$ with $\omega_q'\ =\ (-1)^q\omega_q$. Then

$$d_q\omega_q'+d_{q+1}^*\omega_{q+2}'=(-1)^q(d_q\omega_q+d_{q+1}^*\omega_{q+2})=(-1)^q\lambda\omega_{q+1}=-\lambda\omega_{q+1}' \tag{26}$$

and it follows from (25) that $\widetilde{D}\omega'\ =\ -\lambda\omega'$. This shows that there is a one-to-one correspondence $\omega\leftrightarrow\omega'$ between eigenvectors for $\widetilde{D}$ with eigenvalue λ and eigenvectors with eigenvalue $-\lambda$, and it follows from the definition (10) that $\eta(s\,;\,\widetilde{D})\ =\ 0$ as claimed. (The statement $\eta(s\,;\,T)=\eta(s\,;\,D)$ is similar to [15, proposition(4·20)]).

Finally, as promised, we apply our results to the semiclassical approximation for the partition function of the Chern-Simons gauge theory on 3-manifolds. The partition function of this theory is

$$Z(k)=\int\mathcal{D}A\,e^{ikS(A)}\qquad,\qquad k\in\mathbf{Z} \tag{27}$$

where

$$S(A)=\frac{1}{4\pi}\int_M Tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)\,. \tag{28}$$

The gauge fields A are the $1-$forms on M with values in the Lie algebra of the gauge group $SU(N)$. The parameter k is required to be integer-valued, then the integrand in (27) is gauge-invariant. An expression for the semiclassical approximation for (27) can be obtained from the invariant integration method of A. Schwarz [16, §5]. (We emphasise that Schwarz's method is ideally suited for evaluating the semiclassical approximation for (27). This method leads to the appearance of inverse volume factors $V(H_A)^{-1}$ in the integrand of the expression ((29) below) for the semiclassical approximation (see [16, §5, formula(1)]), where H_A is the subgroup of gauge transformations which leaves the gauge field A unchanged. These factors are necessary to reproduce

the numerical factors appearing in the large k limit of the non-perturbative expression for the partition function and have not been obtained in a self-contained way in other evaluations of the semiclassical approximation for the Chern-Simons partition function[4]. We will be discussing this in more detail in a future paper; see also [17].) The expression obtained from Schwarz's method for the semiclassical approximation for (27) has the form

$$Z_{sc}(k) = \int_{\mathcal{M}^F} \mathcal{D}A\, e^{ikS(A)}\mu(k;A) \tag{29}$$

where $\mathcal{M}^F$ is the moduli space of flat gauge fields modulo gauge transformations. (The flat gauge fields are the solutions to the field equations corresponding to (28)). The integrand $e^{ikS(A)}\mu(k;A)$ is gauge-invariant and is therefore a well-defined function on $\mathcal{M}^F$. The quantity $\mu(k;A)$ is given by [16, §5, formula(1)] and its dependence on k enters through the determinant

$$det(ick\widetilde{T}_A)^{-1/2} \quad , \quad T_A = *d_1^A \tag{30}$$

where c is a numerical constant (involving π) and d_q^A is the flat covariant derivative on the Lie algebra-valued $q-$forms obtained from d_q by "twisting" by the flat gauge field A. (See [18, §15.2] for the definition of this). The results above concerning the map T in (17) generalise for the map T_A in (30). Since in the present case $dimM = 3$, $m = 1$ and it follows from (6) and (23) that the $k-$dependence of the determinant in (30) is given by

$$k^{-\zeta(0\,;\,|*d_1^A|)/2} = k^{(-dimH^0(d^A)+dimH^1(d^A))/2} \,. \tag{31}$$

It follows that in the limit of large k the $k-$dependence of the semiclassical approximation (29) (ignoring phase factors) is given by

$$k^{\left(\max_A \{-dimH^0(d^A)/2+dimH^1(d^A)/2\}\right)} \tag{32}$$

where the maximum is taken over the flat gauge fields. This is precisely the k dependence [9, formula(1.37)] of the large k limit of the partition function (27) obtained

[4]These volume factors were put in by hand in the expression for the semiclassical approximation given by L. Rozansky in [11] and shown to lead to agreement with the large k limit of the non-perturbative expression for the partition function for large classes of 3-manifolds

from non-perturbative calculations[5].

We illustrate this with a specific example. When M is the 3-sphere the expression for the partition function obtained from Witten's non-perturbative method [5, §4] with gauge group $SU(2)$ is

$$Z(k) = \sqrt{\frac{2}{k+2}}\, sin\Big(\frac{\pi}{k+2}\Big) \quad \sim \quad \sqrt{2}\pi k^{-3/2} \quad \text{for} \;\; k \to \infty\,. \tag{33}$$

Since $\pi_1(S^3)$ is trivial the only flat gauge field on the 3-sphere up to gauge equivalence is the trivial field $A\!=\!0$, and in this case we have $dimH^0(d^A)\!=\!dim(su(2))\,dimH^0(S^3)$ $= 3$ and $dimH^1(d^A) \;=\; dim(su(2))\,dimH^1(S^3) \;=\; 0\,.$ It follows from (31) that the $k-$dependence of the semiclassical approximation in this case is $\sim\; k^{-3/2}$, in agreement with the large k limit of (33).

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